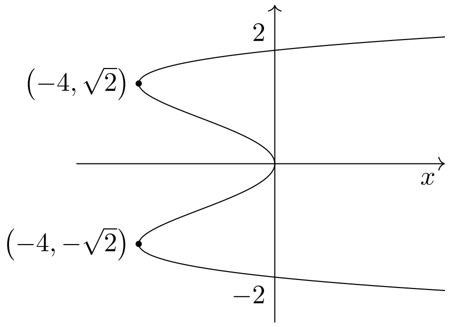
4001. (a) Differentiating, we set $\frac{dx}{dy} = 0$:

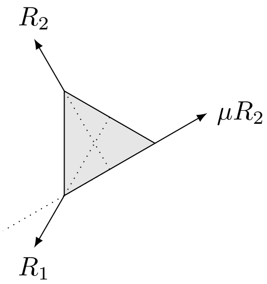
$$\begin{aligned} &4y^3 - 8y = 0 \\ \Longrightarrow\ &y = 0, \pm\sqrt{2}. \end{aligned}$$

Substituting back in, the tangent is parallel to the y axis at $(0,0)$ and $\left(-4, \pm\sqrt{2}\right)$.

(b) The curve is a positive quartic, with x in terms of y. It crosses the y axis at ± 2:



4002. Assuming limiting friction between chock and slope, the forces on the chock are



The chock is in equilibrium. Perpendicular to the slope, $R_2 = R_1 \sin 30°$, so $R_1 = 2R_2$. Parallel to the slope,

$$\begin{aligned} &\mu R_2 - R_1 \cos 30° = 0 \\ \therefore\quad &\mu R_2 - \sqrt{3} R_2 = 0 \\ \therefore\quad &\mu = \sqrt{3}. \end{aligned}$$

For any μ less than this, the chock will slip down the slope, as will the barrel.

4003. The equation holds if either of the factors is zero. The solution curves from the first factor are

$$\begin{aligned} &\frac{dy}{dx} = -2x \\ \Longrightarrow\ &y = -x^2 + c. \end{aligned}$$

The solution curves from the second factor, which is a separable DE with a standard result, are

$$\begin{aligned} &\frac{dy}{dx} = y \\ \Longrightarrow\ &y = Ae^x. \end{aligned}$$

So, any curve with either the form $y = -x^2 + c$ or the form $y = Ae^x$ satisfies the equation.

——— Nota Bene ———

The standard result quoted above can be proved as follows:

$$\begin{aligned} &\frac{dy}{dx} = y \\ \Longrightarrow\ &\int \frac{1}{y}\, dy = \int 1\, dx \\ \Longrightarrow\ &\ln|y| = x + c \\ \Longrightarrow\ &|y| = e^{x+c} \\ \therefore\quad &y = Ae^x. \end{aligned}$$

In the penultimate line, $|y|$ and e^c must both be positive. In the last line, we allow A to take any real value, which permits negative y (such curves as $y = -e^x$ satisfy the penultimate line) as well as the zero solution $y = 0$, which doesn't satisfy the penultimate line, but does satisfy the original differential equation.

4004. If k is triangular, then $k = \frac{1}{2}n(n+1)$, for $n \in \mathbb{N}$:

$$\begin{aligned} 8k + 1 &= 4n(n+1) + 1 \\ &\equiv 4n^2 + 4n + 1 \\ &\equiv (2n+1)^2. \end{aligned}$$

This is a square, proving the forwards implication. Now, let $8k+1$ be a square. Since $8k+1$ is odd, it must be the square of an odd number $2n+1$, where $n \in \mathbb{N}$. This gives

$$\begin{aligned} &8k + 1 = (2n+1)^2 \\ \Longrightarrow\ &8k = 4n^2 + 4n \\ \Longrightarrow\ &k = \tfrac{1}{2}n(n+1). \end{aligned}$$

This proves the backwards implication. So, $k \in \mathbb{N}$ is triangular if and only if $8k+1$ is a square. □

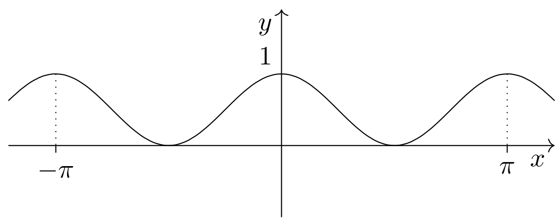
4005. The relevant double-angle identity is

$$\cos 2x \equiv 2\cos^2 x - 1.$$

Using this, we need to sketch

$$y = \tfrac{1}{2}(\cos 2x + 1).$$

This can be viewed as a transformation of the graph $y = \cos x$: a stretch by scale factor $1/2$ in the x direction, and a translation by vector $\mathbf{j}$ followed by a stretch by scale factor $1/2$ in the y direction. This gives



4006. Differentiating (implicitly) with respect to z, we use the chain and product rules:

$$\begin{aligned} & y^2z - y + z = 1 \\ \Longrightarrow\ & 2y\frac{dy}{dz}z + y^2 - \frac{dy}{dz} + 1 = 0 \\ \Longrightarrow\ & \frac{dy}{dz} = \frac{1+y^2}{1-2yz}. \end{aligned}$$

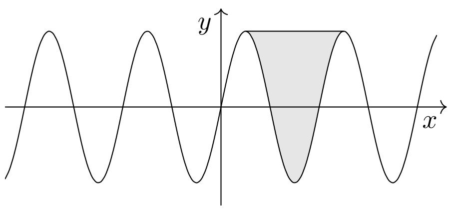
For y to be stationary with respect to z, we need $\frac{dy}{dz} = 0$. But the numerator of the fraction is $1+y^2$, which can never be zero for real y. Hence, y is never stationary with respect to z.

4007. The gradient of the line through these two points is $m = -1/k$. So, the equation of the line is

$$\begin{aligned} & y - k^2 = -\tfrac{1}{k}(x-k) \\ \Longrightarrow\ & y = -\tfrac{1}{k}x + k^2 + 1. \end{aligned}$$

This crosses the y axis at $y = k^2+1$. And, since k is non-zero, k^2 is strictly positive, hence $k^2+1 > 1$ as required.

4008. The curve $y = \sin x$ has consecutive maxima at $x = \pi/2$ and $x = 5\pi/2$. The transformation to $y = \sin 4x$ is a stretch s.f. $1/4$ in the x direction, taking the maxima to $x = \pi/8$ and $x = 5\pi/8$.



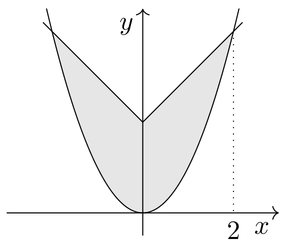
The y value remains 1 under the x stretch. So, the line segment has equation $y = 1$, and we need to calculate the following integral:

$$\begin{aligned} & \int_{\frac{\pi}{8}}^{\frac{5\pi}{8}} 1 - \sin 4x\, dx \\ =\ & \Big[x + \tfrac{1}{4}\cos 4x\Big]_{\frac{\pi}{8}}^{\frac{5\pi}{8}} \\ =\ & \tfrac{5}{8}\pi - \tfrac{1}{8}\pi \\ =\ & \tfrac{1}{2}\pi, \text{ as required.} \end{aligned}$$

4009. Since we are taking the limit as $x \to 0$, we can use small-angle approximations, which become exact in the limit. So,

$$\begin{aligned} & \lim_{x\to 0} \frac{1-\cos 4x}{\sin 3x} \\ =\ & \lim_{x\to 0} \frac{1 - \left(1 - \frac{1}{2}(4x)^2\right)}{3x} \\ =\ & \lim_{x\to 0} \frac{8x^2}{3x} \\ =\ & \lim_{x\to 0} \frac{8x}{3} \\ =\ & 0, \text{ as required.} \end{aligned}$$

4010. The region is as follows. The intersection in the positive quadrant is at $x = 2$:



Using the symmetry of the graphs, the area we are looking for is

$$\begin{aligned} A &= 2\int_0^2 x + 2 - x^2\, dx \\ &= 2\Big[\tfrac{1}{2}x^2 + 2x - \tfrac{1}{3}x^3\Big]_0^2 \\ &= 2\Big(\tfrac{1}{2}\cdot 2^2 + 2\cdot 2 - \tfrac{1}{3}\cdot 2^3 - 0\Big) \\ &= 6\tfrac{2}{3}, \text{ as required.} \end{aligned}$$

4011. The student has lost one value of θ. The first listed value eventually drops out of the domain, when $\pi/2$ is subtracted from line 2 to line 3. But this will bring another value in, which was previously too large. The solution should be

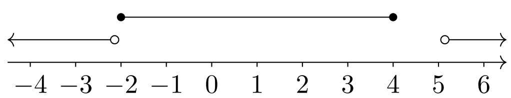
> Solving algebraically,
>
> $$\begin{aligned} & \tan\left(2\theta + \tfrac{\pi}{2}\right) = \sqrt{3} \\ \Longrightarrow\ & 2\theta + \tfrac{\pi}{2} = ..., \tfrac{4\pi}{3}, \tfrac{7\pi}{3}, \tfrac{10\pi}{3}, \tfrac{13\pi}{3}, ... \\ \Longrightarrow\ & \theta = ..., \tfrac{5\pi}{6}, \tfrac{11\pi}{6}, \tfrac{17\pi}{3}, \tfrac{23\pi}{3}, ... \end{aligned}$$
>
> So, in $[0, 2\pi)$, the solution is
>
> $$\theta = \tfrac{5\pi}{6}, \tfrac{11\pi}{6}, \tfrac{17\pi}{3}, \tfrac{23\pi}{6}.$$

4012. The first inequality is $3 \geq |x-1|$: x is a distance no more than 3 from 1, so $x \in [-2, 4]$. Having solved the boundary equation, the second inequality has solution set

$$\left(-\infty, \tfrac{3-\sqrt{53}}{2}\right) \cup \left(\tfrac{3+\sqrt{53}}{2}, \infty\right).$$

The intersection of these two solution sets is empty, however, since

$$\tfrac{3-\sqrt{53}}{2} = -2.14 < -2 \text{ and } \tfrac{3+\sqrt{53}}{2} = 5.14 > 4.$$



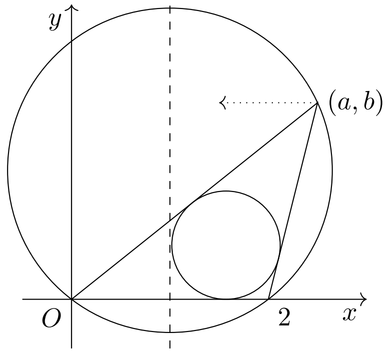
Hence, there are no real numbers x which satisfy both inequalities.

4013. Solving for intersections,

$$\begin{aligned} & a + bx = bx - ax^2 \\ \Longrightarrow\ & a + ax^2 = 0 \\ \Longrightarrow\ & a(1 + x^2) = 0. \end{aligned}$$

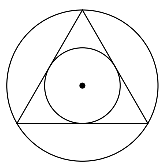
The factor $(1 + x^2)$ can never be zero. So, if the graphs intersect, then a must be zero. This gives both graphs as $y = bx$. Since they are the same straight line, they have infinitely many points of intersection, as required.

4014. Without loss of generality, put two of the vertices at $(0, 0)$ and $(2, 0)$, and the other at (a, b), for $a, b > 0$.



The circumcircle has a centre on the line $x = 1$. Consider the effect of moving point (a, b) in the direction shown, towards the line $x = 1$. Doing so decreases the size of the circumcircle, and increases the size of the incircle. Hence, the ratio of radii is minimised when $a = 1$. By symmetry, the same holds for all sides of the triangle. So, the ratio of radii is minimised when the triangle is equilateral.

For an equilateral triangle, the scenario is



Elementary trigonometry tells us that the ratio of radii is 2. Hence, in general, the ratio of radii of the incircle and the circumcircle is at least 2. □

4015. (a) The probability of $(2, 0, 1)$ is the product of:

$$\begin{aligned} &\mathbb{P}(X = 2) \text{ in } B(3, 1/2), \\ &\mathbb{P}(X = 0) \text{ in } B(1, 1/2), \\ &\mathbb{P}(X = 1) \text{ in } B(1, 1/2). \end{aligned}$$

This gives $\mathbb{P}(2, 0, 1) = \frac{3}{8} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{3}{32}$.

(b) To end after three turns, the last die must be removed on the third turn. The options are

$$\begin{array}{ccc} (2,0,1) & (1,1,1) & (0,2,1) \\ (1,0,2) & (0,1,2) & (0,0,3). \end{array}$$

(c) In each different game, we have the product of three binomial probabilities:

$$\begin{aligned} \mathbb{P}(2,0,1) &= \tfrac{3}{8} \cdot \tfrac{1}{2} \cdot \tfrac{1}{2} = \tfrac{3}{32} \\ \mathbb{P}(1,1,1) &= \tfrac{3}{8} \cdot \tfrac{1}{2} \cdot \tfrac{1}{2} = \tfrac{3}{32} \\ \mathbb{P}(0,2,1) &= \tfrac{1}{8} \cdot \tfrac{3}{8} \cdot \tfrac{1}{2} = \tfrac{3}{128} \\ \mathbb{P}(1,0,2) &= \tfrac{3}{8} \cdot \tfrac{1}{4} \cdot \tfrac{1}{4} = \tfrac{3}{128} \\ \mathbb{P}(0,1,2) &= \tfrac{1}{8} \cdot \tfrac{3}{8} \cdot \tfrac{1}{4} = \tfrac{3}{256} \\ \mathbb{P}(0,0,3) &= \tfrac{1}{8} \cdot \tfrac{1}{8} \cdot \tfrac{1}{8} = \tfrac{1}{512}. \end{aligned}$$

These add to $p = \frac{127}{512}$, as required.

4016. Setting the expression to zero and solving using a calculator, we find real roots at $x = 7$ and $x = -\frac{3}{2}$. Hence, by the factor theorem, the polynomial has factors $(x - 7)$ and $(2x + 3)$. Taking these out, we are left with an irreducible quadratic:

$$\begin{aligned} & 2x^4 - 11x^3 - 19x^2 - 11x - 21 \\ \equiv\ & (x - 7)(2x + 3)(x^2 + 1). \end{aligned}$$

— Alternative Method —

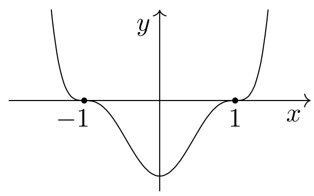
There are roots $x = 7$ and $x = -3/2$, which combine to give a (reducible) quadratic factor of $(2x^2 - 11x - 21)$. Using polynomial long division,

$$\begin{array}{r|l} & x^2 \quad + 0x \quad + 1 \\ \hline 2x^2 - 11x - 21 & 2x^4 - 11x^3 - 19x^2 - 11x - 21 \\ & -\, 2x^4 + 11x^3 + 21x^2 \\ \hline & 0x^3 \quad + 2x^2 - 11x \\ & 0x^3 \quad + 0x^2 \quad + 0x \\ \hline & 2x^2 - 11x - 21 \\ & -\, 2x^2 + 11x + 21 \\ \hline & 0x \quad + 0 \end{array}$$

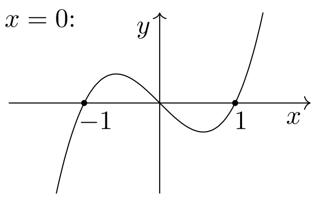
So, the full factorisation is

$$\begin{aligned} & 2x^4 - 11x^3 - 19x^2 - 11x - 21 \\ \equiv\ & (2x^2 - 11x - 21)(x^2 + 1) \\ \equiv\ & (x - 7)(2x + 3)(x^2 + 1). \end{aligned}$$

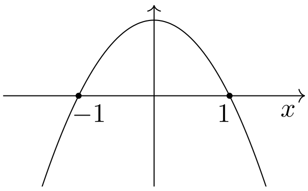
4017. (a) This is a positive sextic graph. In $f(x)\,g(x)$, the multiplicities of the roots of f and g add to give triple roots at $x = \pm 1$:



(b) This is cubic, with roots at $x = \pm 1$. And, since the graphs are symmetrical, it must also have a root at $x = 0$:



(c) Since the functions are monic, the difference is quadratic. It has roots at $x \pm 1$. And, since $\mathrm{f}(0) > 0$ and $\mathrm{g}(0) < 0$, the y intercept is +ve: $\mathrm{f}(0) - \mathrm{g}(0) > 0$. Hence, the quadratic is −ve.



4018. The series is $1 + \frac{1}{2} + \frac{1}{3} + ...$ We group the terms as follows, in sets of $1, 1, 2, 4, 8, ...$ terms:

$$(1) + \left(\tfrac{1}{2}\right) + \left(\tfrac{1}{3} + \tfrac{1}{4}\right) + \left(\tfrac{1}{5} + ... + \tfrac{1}{8}\right) + ...$$

The smallest term in each group is of the form $\frac{1}{2^n}$, and its group has 2^{n-1} terms. This gives a total of more than $\frac{1}{2^n} \times 2^{n-1} = \frac{1}{2}$. Hence, since the sum S_∞ contains an infinite number of such groups, the sum diverges. □

4019. (a) Using two versions of the $\cos 2\theta$ double-angle formula, the RHS is

$$\begin{aligned} &\frac{1 - \cos 2\theta}{1 + \cos 2\theta} \\ \equiv\ &\frac{2\sin^2\theta}{2\cos^2\theta} \\ \equiv\ &\tan^2\theta, \text{ as required.} \end{aligned}$$

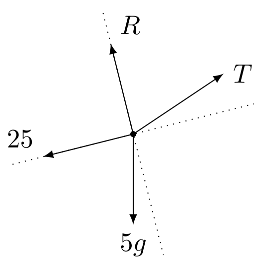
(b) Multiplying top and bottom by $\cos 2\theta$ gives the integrand as

$$\begin{aligned} &\frac{2\sec 2\theta}{\sec 2\theta + 1} \\ \equiv\ &\frac{2}{1 + \cos 2\theta} \\ \equiv\ &\frac{1 + \cos 2\theta + 1 - \cos 2\theta}{1 + \cos 2\theta} \\ \equiv\ &1 + \frac{1 - \cos 2\theta}{1 + \cos 2\theta} \\ \equiv\ &1 + \tan^2\theta \\ \equiv\ &\sec^2\theta. \end{aligned}$$

Hence, we can integrate as follows:

$$\begin{aligned} &\int \frac{2\sec 2\theta}{\sec 2\theta + 1}\, d\theta \\ \equiv\ &\int \sec^2\theta\, d\theta \\ =\ &\tan\theta + c. \end{aligned}$$

4020. (a) Forces:



(b) The sledge is moving at constant velocity so the resultant force is zero. Resolving up the slope, $T\cos 20° - 25 - 5g\sin 10° = 0$. This gives $T = 35.659...$, so $T = 35.7$ N (3sf).

(c) When the child lets go of the string, the sledge is still moving forwards, so the friction remains in the same direction, but the tension drops to zero. Hence, the acceleration is given by

$$a = -\frac{25 + 5g\sin 10°}{5} = -6.701...\ \text{ms}^{-2}.$$

From $u = 1.2$, the distance travelled before the sledge comes to rest is given by

$$0^2 = 1.2^2 - 2 \times 6.701 \times d.$$

This yields $0.1074...$, so $d = 0.107$ m (3sf).

(d) Having come to rest, the frictional force acts up the slope. Assuming that friction is at 25 N, the resultant is $25 - 5g\sin 10° = 16.49...$, acting up the slope. This is a contradiction: friction acts to *oppose* motion.

So, the friction cannot be at its dynamic value of 25 N. Hence, the frictional force will be less than 25 N, and the sledge, having come to rest, will remain at rest.

4021. (a) We solve simultaneously, adding the equations and halving, then subtracting the equations and halving. This gives

$$\begin{aligned} \mathrm{f}(x) &= \tfrac{\sqrt{2}}{2}(\sin x + \cos x), \\ \mathrm{g}(x) &= \tfrac{\sqrt{2}}{2}(\sin x - \cos x). \end{aligned}$$

(b) Simplifying the LHS,

$$\begin{aligned} &\mathrm{f}(x)^2 + \mathrm{g}(x)^2 \\ \equiv\ &\tfrac{1}{2}(\sin^2 x + 2\sin x\cos x + \cos^2 x) \\ &\quad + \tfrac{1}{2}(\sin^2 x - 2\sin x\cos x + \cos^2 x) \\ \equiv\ &\sin^2 x + \cos^2 x \\ \equiv\ &1, \text{ as required.} \end{aligned}$$

4022. If $y = \mathrm{f}(x)$ is a solution curve, then

$$\mathrm{f}'(x) = -4x\,\mathrm{f}(x)^2.$$

Substituting $y = A\,\mathrm{f}(x)$ into the LHS,

$$\begin{aligned} &A\,\mathrm{f}'(x) + 4x\big(A\,\mathrm{f}(x)\big)^2 \\ =\ &A\big(-4x\,\mathrm{f}(x)^2\big) + 4x\big(A\,\mathrm{f}(x)\big)^2 \\ \equiv\ &4x\,\mathrm{f}(x)^2(A^2 - A). \end{aligned}$$

This is only identically equal to zero if $A^2 - A = 0$. This gives $A = 1$, which is the original solution curve, or $A = 0$, which is the trivial solution $y = 0$. So, discounting the latter, the statement is false.

4023. Rearranging to make y^2 the subject,

$$y^2 = \frac{4}{x^2} + x^2$$
$$\Longrightarrow \frac{d}{dx}\left(y^2\right) = -\frac{8}{x^3} + 2x.$$

If y^2 is stationary, then so are both y and $-y$. For SPs, we set the above to zero:

$$-8 + 2x^4 = 0$$
$$\Longrightarrow x = \pm\sqrt{2}.$$

So, the curve has exactly four SPs, at coordinates $(\sqrt{2}, \pm 2)$ and $(-\sqrt{2}, \pm 2)$. These form a rectangle: the ratio of lengths is $1 : \sqrt{2}$.

Bonus: A4, so this very page!

4024. (a) Let $a_n = a + (n-1)c$ and $b_n = b + (n-1)d$. Then the differences are

$$\begin{aligned} &pa_{n+1} + qb_{n+1} - (pa_n + qb_n) \\ &= p(a + nc) + q(b + nd) \\ &\quad - p(a + (n-1)c) - q(b + (n-1)d) \\ &\equiv c + d. \end{aligned}$$

This is constant, so the sequence is arithmetic.

(b) The nth term is k^{a_n}. Hence, the ratio between successive terms is

$$\frac{k^{a_{n+1}}}{k^{a_n}} = \frac{k^{a+nc}}{k^{a+(n-1)c}} \equiv k^c.$$

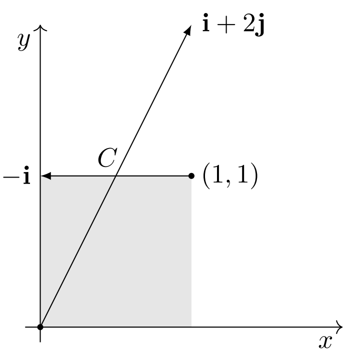
This is constant, so the sequence is geometric.

(c) Using the $(n+1)$th terms for convenience, the relevant limit is

$$\lim_{n\to\infty} \frac{a + nc}{b + nd} \equiv \lim_{n\to\infty} \frac{\frac{a}{n} + c}{\frac{b}{n} + d} \equiv \frac{c}{d}.$$

This is the ratio of common differences.

4025. The first two forces are as follows:



The resultant of these two is $2\mathbf{j}$. So, the third force must be $-2\mathbf{j}$. And, for rotational equilibrium, all three lines of action must pass through the same point, marked C above. Hence, the line of action of the third force is $x = \frac{1}{2}$.

4026. Differentiating by the chain rule,

$$\frac{d}{dx}\left(\frac{1}{\sqrt{r^2 - x^2}}\right) = \frac{x}{(r^2 - x^2)^{\frac{3}{2}}}.$$

Differentiating again, by the quotient rule,

$$\begin{aligned} &\frac{d}{dx}\left(\frac{x}{(r^2 - x^2)^{\frac{3}{2}}}\right) \\ &\equiv \frac{\left(r^2 - x^2\right)^{\frac{3}{2}} - x \cdot -2x \cdot \frac{3}{2}\left(r^2 - x^2\right)^{\frac{1}{2}}}{(r^2 - x^2)^3} \\ &\equiv \frac{\left(r^2 - x^2\right) + 3x^2}{(r^2 - x^2)^{\frac{5}{2}}} \\ &\equiv \frac{r^2 + 2x^2}{(r^2 - x^2)^{\frac{5}{2}}}, \text{ as required.} \end{aligned}$$

4027. There are a number of assumptions required. The three most significant are

① The set of data is large, so we can assume that 25% of the data lie in each quartile.

② The sampling is random (and the set is large), so that successive choices are independent.

③ The data is measured to a high enough degree of accuracy to be modelled continuously, i.e. we don't have to worry about the probability of a datum exactly equalling a quartile value.

Making these assumptions,

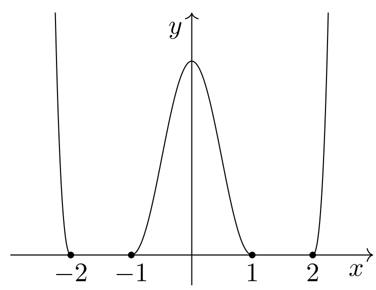
(a) The distribution is $X \sim B(10, 0.25)$, giving

$$\mathbb{P}(X = 0) = 0.75^{10} = 0.0563 \text{ (3sf)}.$$

(b) The distribution is $X \sim B(10, 0.5)$, giving

$$\mathbb{P}(X = 5) = {}^{10}\mathrm{C}_5 \cdot 0.5^5 \cdot 0.5^5 = 0.246 \text{ (3sf)}.$$

4028. The given graph is the square of the required graph. Squaring introduces new solution points. The graph $\sqrt{y} = x^4 - 5x^2 + 4$ has no points where $x^4 - 5x^2 + 4 < 0$. This removes all points in the domain $(-2, -1) \cup (1, 2)$. So, the graph of $\sqrt{y} = x^4 - 5x^2 + 4$ is

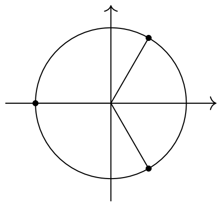


4029. This is true. If a polynomial f has $\mathrm{f}(a) = 0$, then the graph $y = \mathrm{f}(x)$ has an x intercept at $x = a$. Furthermore, since $\mathrm{f}'(a) = 0$, this is a stationary point. Hence, the root at $x = a$ must be (at least) a double root. This implies that $\mathrm{f}(x)$ has a factor of $(x - a)^2$. □

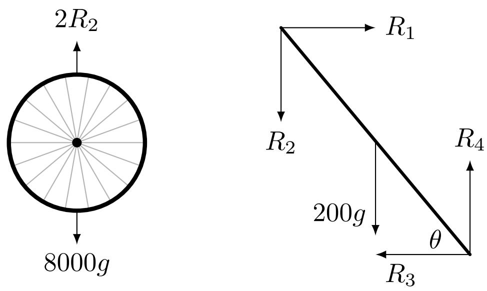
4030. Using the second Pythagorean trig identity,

$$\begin{aligned} & \tan^2\theta - \sec\theta = 1 \\ \Longrightarrow\ & \sec^2\theta - \sec\theta - 2 = 0 \\ \Longrightarrow\ & (\sec\theta + 1)(\sec\theta - 2) = 0 \\ \Longrightarrow\ & \cos\theta = -1, \tfrac{1}{2}. \end{aligned}$$

For $\theta \in [0, 2\pi)$, the solution is $\theta = \frac{\pi}{3}, \pi, \frac{5\pi}{3}$. On a unit circle, these are rotationally symmetrical:



4031. (a) Splitting the contact forces on the strut into horizontal and vertical components, the force diagrams are as follows. R_1 is applied by the other strut, R_2 is applied by the wheel, and R_3 and R_4 are the components of the contact force exerted by the ground.



Using the dimensions of the wheel and struts, the strut is the hypotenuse of a $(3, 4, 5)$ metre triangle. So, $\sin\theta = \frac{4}{5}$ and $\cos\theta = \frac{3}{5}$.

(b) This gives $R_2 = 4000g$. Vertically for the strut, $R_4 = 4200g$. Taking moments around its base,

$$\begin{aligned} & 200g \cdot 2.5\cos\theta + R_2 \cdot 5\cos\theta - R_1 \cdot 5\sin\theta = 0 \\ \Longrightarrow\ & R_1 = 3075g. \end{aligned}$$

So, $R_3 = 3075g$. The contact force exerted on the strut by the ground is

$$\begin{aligned} C &= \sqrt{R_3^2 + R_4^2} \\ &= \sqrt{3075^2g^2 + 4200^2g^2} \\ &= 51012.4... \\ &= 51 \text{ kN (0dp)}. \end{aligned}$$

(c) A spoke/string/cable can only pull, not push. Compression would cause buckling. Hence, it is only the cables below the axle which exert upwards force on the wheel, counteracting the weight. The spokes above the axle can only pull downwards on the wheel. So, tension must be higher in spokes below the axle.

4032. Setting up the gradient,

$$\begin{aligned} & \lim_{k\to 1} \frac{(k^2+1) - 2k}{2k - (k+1)} \\ =\ & \lim_{k\to 1} \frac{(k-1)^2}{k-1} \\ =\ & \lim_{k\to 1} k - 1 \\ =\ & 0, \text{ as required.} \end{aligned}$$

4033. Multiplying by $\sqrt{x}$,

$$\begin{aligned} & x^3 + x^{\frac{3}{2}} - 756 = 0 \\ \Longrightarrow\ & x^{\frac{3}{2}} = -28, 27. \end{aligned}$$

We reject -28, as the denominator of the index is even. This gives $x^{\frac{3}{2}} = 27$, so $x = 9$.

4034. (a) This is $\mathbb{P}(X_1 = 1, 3)$. By symmetry, the two individual probabilities are equal:

$$\mathbb{P}(|X_1 - 2| = 1) = 2\,\mathbb{P}(X_1 = 1) = \tfrac{1}{2}.$$

(b) We square the probabilities of $X \sim B(4, 0.5)$, and add them:

$$\begin{aligned} & \mathbb{P}(X_1 = X_2) \\ =\ & \tfrac{1}{16^2}\left(1^2 + 4^2 + 6^2 + 4^2 + 1^2\right) \\ =\ & \tfrac{35}{128}. \end{aligned}$$

4035. The integrand is of the standard form $\mathrm{f}'(x)/\mathrm{f}(x)$, with $\mathrm{f}(x) = \ln x$. So, we integrate by inspection, using the reverse chain rule:

$$\int \frac{1}{x\ln x}\,dx = \ln|\ln x| + c.$$

4036. The first equation is

$$\begin{aligned} & \sin y = \sqrt{3}\sin x \\ \Longrightarrow\ & 1 - \sin^2 y = 1 - 3\sin^2 x \\ \Longrightarrow\ & \cos^2 y = 3\cos^2 x - 2. \end{aligned}$$

The second equation is

$$\begin{aligned} & \cos y = \sqrt{3}\cos x - 2 \\ \Longrightarrow\ & \cos^2 y = 3\cos^2 x - 4\sqrt{3}\cos x + 4. \end{aligned}$$

Subtracting the two,

$$\begin{aligned} & 0 = 4\sqrt{3}\cos x - 6 \\ \Longrightarrow\ & \cos x = \tfrac{\sqrt{3}}{2} \\ \Longrightarrow\ & x = \tfrac{\pi}{6}, ... \end{aligned}$$

Substituting into the first equation, this gives $\sin y = \sqrt{3}/2$, so $y = \pi/3, 2\pi/3$. Substituting into the second equation, $\cos y = -1/2$, so $y = 2\pi/3$. Hence, the only (x, y) solution point with x and y both in $[0, \pi)$ is $(\pi/6, 2\pi/3)$.

4037. Each such hexagon passes through the centre of the cube, and is normal to one of the cube's space diagonals. There are four space diagonals, one for each of the vertices of e.g. the base. Hence, there are four hexagons. The possibility space consists of ${}^{12}\mathrm{C}_6 = 924$ outcomes. This gives

$$p = \tfrac{4}{924} = \tfrac{1}{231}, \text{ as required.}$$

4038. (a) Differentiating implicitly,

$$\begin{aligned} &e^x + e^y = 2 \\ \Longrightarrow\ &e^x + e^y \tfrac{dy}{dx} = 0. \end{aligned}$$

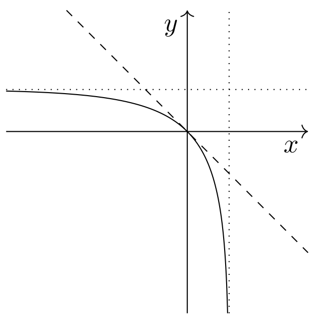
Setting $\frac{dy}{dx} = -1$, $e^x = e^y$. Substituting this back into the equation of the curve, $2e^x = 2$, which gives $x = 0$, $y = 0$. Hence, line $y = -x$ is tangent to the curve at the origin.

(b) Rearranging to make y the subject,

$$\begin{aligned} &e^y = 2 - e^x \\ \Longrightarrow\ &y = \ln(2 - e^x). \end{aligned}$$

As $x \to \ln 2$, $2 - e^x \to 0$, and $y \to -\infty$. Hence, $x = \ln 2$ is an asymptote. Since the original equation is symmetrical in x and y, there is also an asymptote at $y = \ln 2$.

(c) Setting $x = 0$, $1 + e^y = 2$, so $y = 0$. Hence, the only axis intercept is the origin. The curve is symmetrical in $y = x$. With the tangent line $y = -x$ dashed and the asymptotes dotted, the curve is



4039. The fraction may be rewritten

$$\frac{\cos\theta}{\cos\theta + 1} \equiv \frac{\cos\theta + 1 - 1}{\cos\theta + 1} \equiv 1 - \frac{1}{\cos\theta + 1}.$$

Since θ is small, $\cos\theta = 1 - \frac{1}{2}\theta^2$. The fraction above, using the generalised binomial expansion, is approximately

$$\begin{aligned} &\frac{1}{(1 - \frac{1}{2}\theta^2) + 1} \\ \equiv\ &\left(2 - \tfrac{1}{2}\theta^2\right)^{-1} \\ \equiv\ &\tfrac{1}{2}\left(1 - \tfrac{1}{4}\theta^2\right)^{-1} \\ \equiv\ &\tfrac{1}{2}\left(1 + \tfrac{1}{4}\theta^2 + ...\right) \\ \approx\ &\tfrac{1}{2} + \tfrac{1}{8}\theta^2. \end{aligned}$$

So, $\dfrac{\cos\theta}{\cos\theta + 1} \approx \frac{1}{2} - \frac{1}{8}\theta^2$, as required.

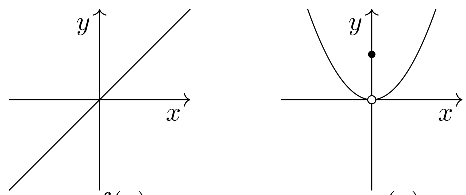
4040. (a) This is true.

If $\mathrm{f}(x) = 0$, then the only way $\mathrm{f}(x)/\mathrm{g}(x)$ could be non-zero is if $\mathrm{g}(x)$ is zero, rendering $\mathrm{f}(x)/\mathrm{g}(x)$ undefined. But 0 is not in the range of g, so the implication holds.

(b) This is not true.

As a counterexample, let $\mathrm{f}(x) = x$ and define g as follows:

$$\mathrm{g} : \begin{cases} x \longmapsto 1, & x = 0, \\ x \longmapsto x^2, & x \neq 0. \end{cases}$$



$y = \mathrm{f}(x)$ $y = \mathrm{g}(x)$

The functions f and g are both defined over $\mathbb{R}$, and 0 is not in the range of g. And

$$\lim_{x\to 0} \mathrm{f}(x) = \lim_{x\to 0} x = 0.$$

But the second statement doesn't follow:

$$\lim_{x\to 0} \frac{\mathrm{f}(x)}{\mathrm{g}(x)} = \lim_{x\to 0} \frac{1}{x}.$$

This limit diverges, disproving the result.

4041. The possibility space consists of 216 equally likely outcomes, in a $6 \times 6 \times 6$ cube. We know that $X + Y + Z = 6$. So, the restricted possibility space for X and Y, classified by the value of Z, consists of the following numbers of outcomes:

Z	1	2	3	4
Total	16	9	4	1
Successful	0	1	2	3.

So, the probability is

$$\begin{aligned} &\mathbb{P}\big(X + Y = Z \mid X + Y + Z = 6\big) \\ =\ &\frac{0 + 1 + 2 + 3}{16 + 9 + 4 + 1} \\ =\ &\tfrac{1}{5}. \end{aligned}$$

4042. Using the second Pythagorean trig identity,

$$\begin{aligned} &\int \tan^2 x\, dx \\ =\ &\int \sec^2 x - 1\, dx \\ =\ &\tan x - x + c. \end{aligned}$$

4043. The derivative of a quartic is a cubic. Since there are stationary points at $x = 2, 3, 4$, this cubic must be of the form

$$\begin{aligned}\frac{dy}{dx} &= k(x-2)(x-3)(x-4)\\ &\equiv k(x^3 - 9x^2 + 26x - 24).\end{aligned}$$

Integrating the above,

$$y = k\big(\tfrac{1}{4}x^4 - 3x^3 + 13x^2 - 24x\big) + c.$$

Since the curve passes through the origin, we know $c = 0$. The point $(1, -55)$ gives

$$\begin{aligned}-55 &= k\big(\tfrac{1}{4} - 3 + 13 - 24\big)\\ \Longrightarrow k &= 4.\end{aligned}$$

So, the equation of the curve is

$$y = x^4 - 12x^3 + 52x^2 - 96x.$$

4044. (a) Solving for intersections,

$$\begin{aligned}t^4 - t^3 &= 20t - 32\\ \Longrightarrow t^4 - t^3 - 20t + 32 &= 0.\end{aligned}$$

Using a polynomial solver, $t = 2$. At this time, the particles have $x_1 = x_2 = 8$.

(b) Differentiating, the velocities are

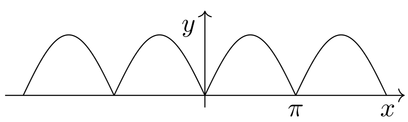
$$\begin{aligned}\dot{x}_1 &= 4t^3 - 3t^2,\\ \dot{x}_2 &= 20.\end{aligned}$$

So, the relative velocity is $4t^3 - 3t^2 - 20$. At the moment of collision,

$$4t^3 - 3t^2 - 20\,\Big|_{t=2} = 0.$$

The relative speed at the moment of collision is therefore instantaneously zero.

4045. The graph of $y = |\sin\theta|$ is



This has period π, so the average value on $\mathbb{R}$ is the same as the average value on $[0, \pi]$. Over this domain, $|\sin\theta| = \sin\theta$. The domain has length π, so the average value is given by

$$\frac{1}{\pi}\int_0^\pi \sin\theta\,d\theta = \frac{1}{\pi}\Big[-\cos\theta\Big]_0^\pi = \frac{2}{\pi}.$$

— Nota Bene —

The general formula for the average value $\overline{\mathrm{f}(x)}$ of a function f, over the domain $[a, b]$, is

$$\overline{\mathrm{f}(x)} = \frac{1}{b-a}\int_a^b \mathrm{f}(x)\,dx.$$

4046. The factor theorem tells us that

$$\mathrm{g}(x) = \big(x-1\big)\big(x-k\big)\big(x-k^2\big).$$

Multiplying out and equating coefficients of x^2,

$$p = 1 + k + k^2.$$

The RHS is the sum of a geometric progression with first term 1, common ratio k and $n = 3$ terms. Quoting the formula,

$$p = \frac{1-k^3}{1-k}, \text{ as required.}$$

4047. (a) The cosec function is undefined at $x = 0$. So, there is no y intercept. For x intercepts,

$$\begin{aligned}&\operatorname{cosec}^2 x + \sin x + 3 = 0\\ \Longrightarrow\ &\sin^3 x + 3\sin^2 x + 1 = 0\\ \Longrightarrow\ &\sin x = -3.104.\end{aligned}$$

This is outside the range of the sine function, so there are no axis intercepts.

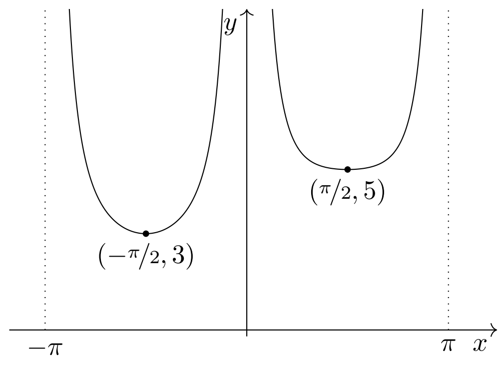
For stationary points,

$$\begin{aligned}&\tfrac{d}{dx}\big((\sin x)^{-2} + \sin x + 3\big) = 0\\ \Longrightarrow\ & -2(\sin x)^{-3}\cos x + \cos x = 0\\ \Longrightarrow\ & \cos x(-2 + \sin^3 x) = 0.\end{aligned}$$

The latter factor has no roots, as the range of $\sin^3$ is $[-1, 1]$. So, $x = -\pi/2, \pi/2$. This gives SPs at $(-\pi/2, 3)$ and $(\pi/2, 5)$.

There is an asymptote at $x = 0$. There are also asymptotes at the boundary of (although outside) the domain, at $x = \pm\pi$.

(b) Collating the above information, the curve is



4048. Using a polynomial solver,

$$\begin{aligned}&3003x^3 + 1896x^2 + 51x - 90 = 0\\ \Longrightarrow\ &x = -\tfrac{3}{7}, -\tfrac{5}{13}, \tfrac{2}{11}.\end{aligned}$$

The factor theorem gives factors $(7x+3)$, $(13x+5)$ and $(11x-2)$. Checking the leading coefficients, $7 \times 11 \times 13 = 1001$, so we need a constant factor of 3. Hence, the original expression factorises as

$$\begin{aligned}&3003x^3 + 1896x^2y + 51xy^2 - 90y^3\\ \equiv\ &3(7x+3y)(13x+5y)(11x-2y).\end{aligned}$$

4049. Let $u = x^4 - 1$, so that $du = 4x^3 dx$. Using this, we can write

$$2x^7\,dx = \tfrac{1}{2}x^4 \cdot 4x^3\,dx = \tfrac{1}{2}(u+1)\,du.$$

Enacting the substitution,

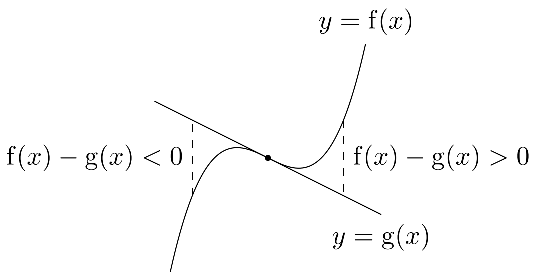
$$\begin{aligned}
&\int \frac{2x^7}{(x^4-1)^{\frac{3}{2}}}\,dx \\
&= \tfrac{1}{2}\int \frac{u+1}{u^{\frac{3}{2}}}\,du \\
&\equiv \tfrac{1}{2}\int u^{-\frac{1}{2}} + u^{-\frac{3}{2}}\,du \\
&= u^{\frac{1}{2}} - u^{-\frac{1}{2}} + c \\
&\equiv \frac{u-1}{\sqrt{u}} + c.
\end{aligned}$$

Written in terms of x, this is $\dfrac{x^4-2}{\sqrt{x^4-1}} + c$.

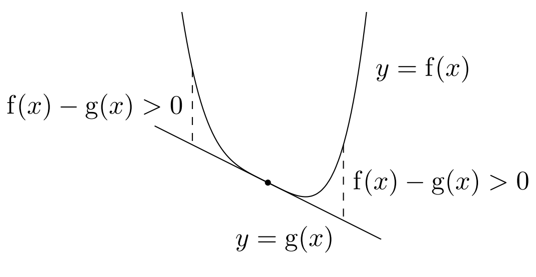
4050. The equation for intersections of the straight line and the quartic is $\text{f}(x) - \text{g}(x) = 0$. This has two distinct roots, each of which is a repeated root. Hence, since a quartic can have a maximum of four roots, each must be a double root. So, there are no points at which $y = \text{g}(x)$ crosses $y = \text{f}(x)$. Since the quartic is positive, $\text{g}(x) \geq \text{f}(x)$ for all x. Hence, $\text{f}(x) - \text{g}(x) \leq 0$ for all $x \in \mathbb{R}$. □

— Nota Bene —

The relevant fact here is that polynomial curves $y = \text{f}(x)$ and $y = \text{g}(x)$ only cross at $x = \alpha$ if the equation $\text{f}(x) - \text{g}(x) = 0$ has a root of odd multiplicity at $x = \alpha$. This is because "crossing" requires a sign change in $\text{f}(x) - \text{g}(x)$.



The above shows a triple root in $\text{f}(x) - \text{g}(x) = 0$, the equation for intersections: there is a sign change. The below shows a quadruple root in the equation for intersections: there is no sign change.



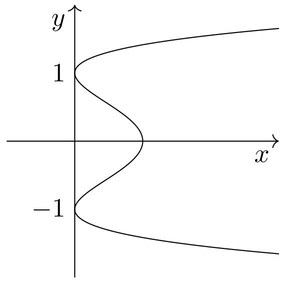
4051. Using the compound-angle formulae,

$$\begin{aligned}
&\cot(\alpha \pm \beta) \\
&\equiv \frac{\cos\alpha\cos\beta \mp \sin\alpha\sin\beta}{\sin\alpha\cos\beta \pm \cos\alpha\sin\beta} \\
&\equiv \frac{\frac{\cos\alpha\cos\beta}{\sin\alpha\sin\beta} \mp 1}{\frac{\cos\beta}{\sin\beta} \pm \frac{\cos\alpha}{\sin\alpha}} \\
&\equiv \frac{\cot\alpha\cot\beta \mp 1}{\cot\beta \pm \cot\alpha}, \quad \text{as required.}
\end{aligned}$$

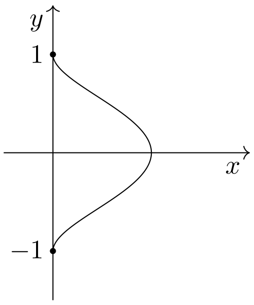
4052. Rearranging and squaring,

$$\begin{aligned}
&\sqrt{x} + y^2 = 1 \\
\implies &x = (1+y)^2(1-y)^2.
\end{aligned}$$

This is a positive quartic, which is tangent to the y axis at $y = \pm 1$. Its graph is



However, the original graph requires $\sqrt{x} \geq 0$. This is $1 - y^2 \geq 0$, which is $y \in [-1, 1]$. So, the original graph is



4053. The relevant inequality is $10X - X^2 > 16$, which has solution $X \in (2, 8)$. The possibility space is $\{X \in \mathbb{R} : 0 \leq X \leq 10\}$, which has length 10. The interval $(2, 8)$ has length 6 (the endpoints have zero probability). So, $\mathbb{P}(Y > 16) = \frac{6}{10} = 0.6$.

4054. The mean of x_1 and x_2 is half of their sum.

$$\begin{aligned}
x_1 + x_2 &= \frac{\sqrt[3]{a}+1}{\sqrt[3]{a}-1} + \frac{\sqrt[3]{a}-1}{\sqrt[3]{a}+1} \\
&\equiv \frac{(\sqrt[3]{a}+1)^2 + (\sqrt[3]{a}-1)^2}{(\sqrt[3]{a}+1)(\sqrt[3]{a}-1)} \\
&\equiv \frac{\sqrt[3]{a^2} + 2\sqrt[3]{a} + 1 + \sqrt[3]{a^2} - 2\sqrt[3]{a} + 1}{\sqrt[3]{a^2} - 1} \\
&\equiv \frac{2\sqrt[3]{a^2} + 2}{\sqrt[3]{a^2} - 1}.
\end{aligned}$$

Dividing by two, $\bar{x} = \dfrac{\sqrt[3]{a^2}+1}{\sqrt[3]{a^2}-1}$, as required.

4055. The prime factorisation of 120 is $2^3 \times 3 \times 5$. A set of five consecutive integers clearly contains a multiple of 3 and a multiple of 5. Furthermore, the set must contain at least two consecutive even numbers. Since these are consecutive, at least one of these must be a multiple of 4. This provides at least three factors of 2. Hence, the product is divisible by $2^3 \times 3 \times 5 = 120$. QED.

4056. (a) $\mathrm{P}(0) = 3$, so the initial population is 3000.

(b) The range of the sine function is $[-1, 1]$. So, the range of $0.4 \sin 3.1t$ is $[-0.4, -0.4]$. The rest of the function is linear and increasing, with value $3 + 0.3 \times \frac{26}{3} = 5.6$ at $t = \frac{26}{3}$. So, the population cannot exceed 6000 before this.

(c) For SPs,

$$\begin{aligned} 0.3 + 0.4 \cdot 3.1 \cos 3.1t &= 0 \\ \Longrightarrow \cos 3.1t &= -\tfrac{15}{62}. \end{aligned}$$

The relevant values $t \geq \frac{26}{3}$ are

$$\begin{aligned} t &= \frac{8\pi + \arccos\left(-\frac{15}{62}\right)}{3.1} = 8.69287, \\ t &= \frac{10\pi + \arccos\left(-\frac{15}{62}\right)}{3.1} = 10.9717. \end{aligned}$$

These give, to 4sf,

$$\begin{aligned} \mathrm{P}(8.69287) &= 5.996, \\ \mathrm{P}(10.9717) &= 6.499. \end{aligned}$$

So, the population falls just short of 6000 at the first of these maxima, but exceeds it at the second.

(d) The relevant equation is

$$\begin{aligned} 6 &= 3 + 0.3t + 0.4 \sin 3.1t \\ \Longrightarrow 0.3t + 0.4 \sin 3.1t - 3 &= 0. \end{aligned}$$

The Newton-Raphson iteration is

$$t_{n+1} = t_n - \frac{0.3t_n + 0.4 \sin 3.1t_n - 3}{0.3 + 1.24 \cos 3.1t_n}.$$

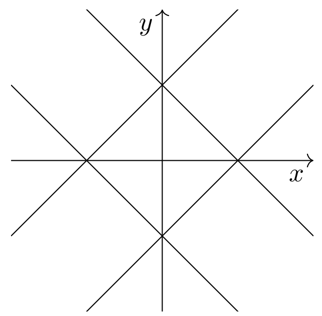
Running this iteration with $t_0 = 10$, we get $t_1 = 10.112...$ and then $t_n \to 10.1080....$ So, to 3sf, the population first exceeds 6000 after 10.11 years (2dp).

4057. Simplifying the product,

$$\begin{aligned} P &= \prod_{i=2}^{\infty} \left(1 - \frac{1}{i^2}\right) \\ &= \prod_{i=2}^{\infty} \frac{i^2 - 1}{i^2} \\ &= \prod_{i=2}^{\infty} \frac{(i-1)(i+1)}{i^2} \\ &= \frac{1 \cdot 3}{2^2} \times \frac{2 \cdot 4}{3^2} \times \frac{3 \cdot 5}{4^2} \times \frac{4 \cdot 6}{5^2} \times ... \end{aligned}$$

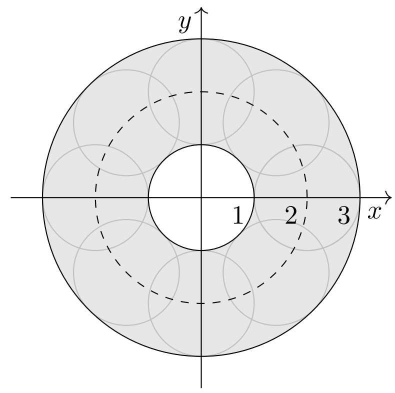
Consider e.g. 3^2 in the second denominator. There are single factors of 3 in the numerators of each of the adjacent fractions. These cancel to give 1. The same cancellation occurs for all denominators (and numerators) thereafter. In the infinite limit, the only factor left uncancelled is one factor of 2 in the first denominator. Therefore, $P = \frac{1}{2}$.

4058. From the first equation $x + y = \pm 1$, and from the second equation $x - y = \pm 1$. The $\pm$ signs are independent. Sketching these,



There are four solutions: $(\pm 1, 0)$ and $(0, \pm 1)$.

4059. Consider $(2 \cos \theta, 2 \sin \theta)$ as a parametric graph in its own right. It is a circle, radius 2, centred at the origin. Call it C_{centre} (dashed below). The set of points which lie on at least one of the family of circles is the set of points which lie a distance of 1 from C_{centre}. With some example circles, this is



4060. Differentiating implicitly,

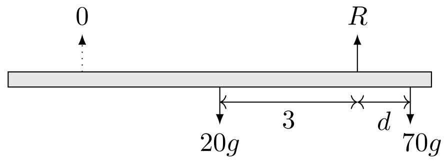
$$\begin{aligned} 2x^2 + 5xy + 2y^2 &= 1 \\ \Longrightarrow 4x + 5y + 5x\tfrac{dy}{dx} + 4y\tfrac{dy}{dx} &= 0. \end{aligned}$$

Setting $\frac{dy}{dx} = 0$ for SPs, $4x + 5y = 0$. Substituting this into the equation of the curve,

$$\begin{aligned} 2x^2 + 5x\left(-\tfrac{4}{5}x\right) + 2\left(-\tfrac{4}{5}x\right)^2 &= 1 \\ \Longrightarrow x^2 + \tfrac{18}{25} &= 0. \end{aligned}$$

This has no real roots. Hence, the graph has no stationary points, as required.

4061. Let the beam be on the point of tipping around the RH support. The reaction at the LH support is therefore zero. Let the distance of the person from the right-hand support be d. The forces are



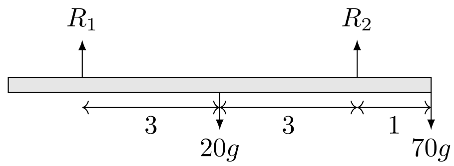
Taking moments around the right-hand support,

$$70g \times d = 20g \times 3$$
$$\implies d = \tfrac{6}{7} < 1.$$

So, if the person stands in the last $\frac{1}{7}$ of a metre at either end, the beam will tip.

—— Alternative Method ——

Put the person at the end of the beam, and assume that it is in equilibrium. The force diagram is



Taking moments around the right-hand support,

$$6R_1 - 3 \cdot 20g + 70g = 0$$
$$\implies R_1 = -10g.$$

A support cannot produce such a negative reaction force, so the beam cannot be equilibrium. There are regions of the beam on which a 70kg person cannot stand without tipping it.

4062. (a) Differentiating by the chain rule,

$$y = \ln\big(\tan(x + \tfrac{\pi}{4})\big)$$
$$\implies \frac{dy}{dx} = \frac{\sec^2(x + \frac{\pi}{4})}{\tan(x + \frac{\pi}{4})}$$

Using $1 + \tan^2\theta \equiv \sec^2\theta$, this is

$$\frac{1 + \tan^2(x + \frac{\pi}{4})}{\tan(x + \frac{\pi}{4})}$$
$$\equiv \cot(x + \tfrac{\pi}{4}) + \tan(x + \tfrac{\pi}{4})$$

(b) Expanding the second term in the above with a compound-angle formula,

$$\tan(x + \tfrac{\pi}{4}) \equiv \frac{\tan x + 1}{1 - \tan x}.$$

Reciprocating this,

$$\cot(x + \tfrac{\pi}{4}) \equiv \frac{1 - \tan x}{\tan x + 1}.$$

Adding these two over a common denominator, the derivative $\frac{dy}{dx}$ is equal to

$$\cot(x + \tfrac{\pi}{4}) + \tan(x + \tfrac{\pi}{4})$$
$$\equiv \frac{\tan^2 x + 2\tan x + 1 + \tan^2 - 2\tan x + 1}{1 - \tan^2 x}$$
$$\equiv \frac{2\tan^2 x + 2}{1 - \tan^2 x}.$$

(c) Using a double-angle formula,

$$\begin{aligned} 2\sec 2x &\equiv \frac{2}{\cos 2x} \\ &\equiv \frac{2}{\cos^2 x - \sin^2 x} \\ &\equiv \frac{2\sec^2 x}{1 - \tan^2 x} \\ &\equiv \frac{2\tan^2 x + 2}{1 - \tan^2 x}. \end{aligned}$$

With part (b), this proves the result.

4063. (a) Solving for intersections,

$$x^2 - x = x^3 - 3x^2 + 2x$$
$$\implies x = 0, 1, 3.$$

The derivatives are

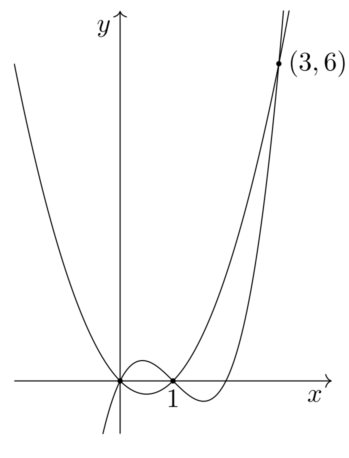
$$\frac{dy}{dx} = 2x - 1,$$
$$\frac{dy}{dx} = 3x^2 - 6x + 2.$$

So, at the intersections, the gradients are

x	0	1	3
$2x - 1$	-1	1	5
$3x^2 - 6x + 2$	2	-1	11

At $x = 1$, the gradients 1 and -1 satisfy $m_1 m_2 = -1$, so the curves are normal to each other at this point.

(b) The coordinates of the intersections are $(0, 0)$, $(1, 0)$ and $(3, 6)$. The graphs are a positive quadratic and a positive cubic:



4064. Since f is even, the integral in question is twice the integral from 0 to ∞. Writing this as a limit,

$$\begin{aligned}&\lim_{k\to\infty} 2\int_0^k e^{1-2x}\,dx\\ &= \lim_{k\to\infty} 2\left[-\tfrac{1}{2}e^{1-2x}\right]_0^k\\ &= \lim_{k\to\infty} 2\left(-\tfrac{1}{2}e^{1-2k} - (-\tfrac{1}{2}e)\right)\\ &= \lim_{k\to\infty}\left(e - e^{1-2k}\right).\end{aligned}$$

As $k \to \infty$, the exponential $e^{1-2k} \to 0$. So,

$$\int_{-\infty}^{\infty} \mathrm{f}(x)\,dx = e.$$

4065. The product S_1S_2 is zero if either or both of S_1 and S_2 is zero. So, we can rewrite as

$$\mathbb{P}(S_1 = 1 \text{ and } S_2 = 1) = 1 - \tfrac{19}{55}.$$

Converting this into algebra,

$$\begin{aligned}&\frac{100-n}{100}\times\frac{99-n}{99} = \frac{36}{55}\\ \Longrightarrow\ & n = 19, 180.\end{aligned}$$

Since $180 > 100$, the solution is $n = 19$.

4066. (a) Using the product rule, the derivatives are

$$\begin{aligned}&\theta = 3e^{4t} + 5e^t\cos 2t\\ \Longrightarrow\ &\tfrac{d\theta}{dt} = 12e^{4t} + 5e^t\cos 2t - 10e^t\sin 2t\\ \Longrightarrow\ &\tfrac{d^2\theta}{dt^2} = 48e^{4t} + 5e^t\cos 2t - 10e^t\sin 2t\\ &\qquad - 10e^t\sin 2t - 20e^t\cos 2t\\ &\quad = 48e^{4t} - 15e^t\cos 2t - 20e^t\sin 2t.\end{aligned}$$

Substituting these into the LHS of the DE,

$$\begin{aligned}&48e^{4t} - 15e^t\cos 2t - 20e^t\sin 2t\\ &\quad - 2(12e^{4t} + 5e^t\cos 2t - 10e^t\sin 2t)\\ &\quad + 5(3e^{4t} + 5e^t\cos 2t)\\ &\equiv 39e^{4t}.\end{aligned}$$

So, the solution satisfies the DE.

(b) As $t \to \infty$, the exponentials grow without bound. The chemist is modelling a physical (as opposed to abstract mathematical) process, in which such unlimited growth is impossible. So, long-term behaviour cannot be as the model suggests.

4067. Multiplying up by the denominators,

$$\begin{aligned}&\frac{\sin x}{1-\cos x} + \frac{\cos x}{1-\sin x} = 0\\ \Longrightarrow\ &\sin x(1-\sin x) + \cos x(1+\cos x) = 0\\ \Longrightarrow\ &\sin x - \sin^2 x + \cos x + \cos^2 x = 0\\ \Longrightarrow\ &\sin x + \cos x = 1.\end{aligned}$$

Writing the LHS in harmonic form,

$$\begin{aligned}&\sqrt{2}\sin\left(x + \tfrac{\pi}{4}\right) = 1\\ \Longrightarrow\ &\sin\left(x + \tfrac{\pi}{4}\right) = \tfrac{1}{\sqrt{2}}\\ \Longrightarrow\ &x + \tfrac{\pi}{4} = \tfrac{\pi}{4}, \tfrac{3\pi}{4}, \ldots\\ \Longrightarrow\ &x = 0, \tfrac{\pi}{2}, \ldots\end{aligned}$$

But the original fractions are undefined at these values, since $1-\cos x = 0$ at $x = 0$ and $1-\sin x = 0$ at $x = \frac{\pi}{2}$. The same is true for subsequent roots. Hence, the solution set is empty.

— Nota Bene —

The procedure for writing in harmonic form is as follows. I'm laying this out explicitly here because, unlike in much mathematics, the *decision-making* logic isn't easily gleaned from a worked solution. And you can save yourself much work here by choosing wisely.

$$\boxed{\sin x + \cos x}$$

Firstly, choose the form (if it isn't given to you). The most reliable method, which will work for any example, is to choose the trig function to match the first term (which we assume is positive), and then to choose the $\pm$ (with reference to compound-angle formulae) to match the sign of the second term.

So, choosing sin, the relevant identity is

$$\begin{aligned}R\sin(x+\alpha) &\equiv R\sin x\cos\alpha + R\cos x\sin\alpha\\ &\equiv 1\sin x + 1\cos x.\end{aligned}$$

Equating coefficients,

$$\begin{aligned}\sin x &: R\cos\alpha = 1,\\ \cos x &: R\sin\alpha = 1.\end{aligned}$$

Because we have made sure to choose the easiest formula (matching all $\pm$ signs), we can quote the primary value for R and the primary value for α without needing to substitute either back in.

- Squaring the equations and adding them, R is the Pythagorean sum of the values on the RHS. So, $R = \sqrt{1^2 + 1^2} = \sqrt{2}$.
- Dividing the bottom equation by the top, $\tan\alpha = 1$. So, $\alpha = \frac{\pi}{4}$.

Due to the initial setup, these values automatically match each other, giving

$$\sin x + \cos x \equiv \sqrt{2}\sin\left(x + \tfrac{\pi}{4}\right).$$

4068. Using the quotient rule, the derivative is

$$\begin{aligned} f'(x) &= \frac{2x(x^2+x+2)-(x^2+1)(2x+1)}{(x^2+x+2)^2} \\ &\equiv \frac{x^2+2x-1}{(x^2+x+2)^2}. \end{aligned}$$

For stationary points, $x^2+2x-1=0$, which has solution $x=-1\pm\sqrt{2}$. At the first SP,

$$\begin{aligned} a &= f\left(-1-\sqrt{2}\right) \\ &= \frac{\left(-1-\sqrt{2}\right)^2+1}{\left(-1-\sqrt{2}\right)^2+\left(-1-\sqrt{2}\right)+2} \\ &= \frac{4+2\sqrt{2}}{4+\sqrt{2}} \\ &= \frac{6+2\sqrt{2}}{7}. \end{aligned}$$

At the second stationary point,

$$\begin{aligned} b &= f\left(-1+\sqrt{2}\right) \\ &= \frac{\left(-1+\sqrt{2}\right)^2+1}{\left(-1+\sqrt{2}\right)^2+\left(-1+\sqrt{2}\right)+2} \\ &= \frac{4-2\sqrt{2}}{4-\sqrt{2}} \\ &= \frac{6-2\sqrt{2}}{7}. \end{aligned}$$

Subtracting these, $a-b=\dfrac{4\sqrt{2}}{7}$, as required.

4069. The implication is backwards. Assume the second statement, and let G be an indefinite integral of g, so that $G'(x)\equiv g(x)$:

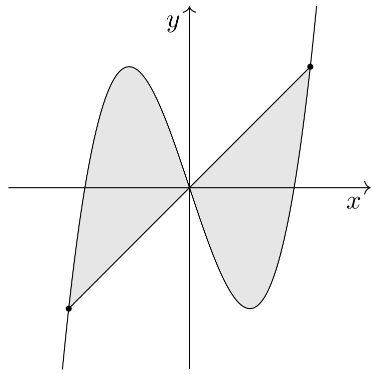
$$\begin{aligned} f(x) &\equiv \int_0^x g(t)\,dt \\ &\equiv \Big[G(t)\Big]_0^x \\ &\equiv G(x)-G(0). \end{aligned}$$

Differentiating gives $f'(x)\equiv g(x)$.

A counterexample to the forwards implication is $f(x)=x^2+1$, $g(x)=2x$. The first statement holds. But the second doesn't:

$$\begin{aligned} \int_0^x 2t\,dt &\equiv \Big[t^2\Big]_0^x \\ &\equiv x^2 \\ &\not\equiv f(x). \end{aligned}$$

4070. The equation $y=x^3+kx$ contains only odd powers of x. Hence, the curve has odd symmetry, which is rotational symmetry around the origin. So, the two points $x=\pm p$ are images of one another under this symmetry. This means the chord in question goes through the origin, and that the two areas in question are rotationally symmetrical.



Hence, their areas are equal. QED.

— Alternative Method —

The chord has gradient p^2+k. It passes through the origin, so has equation

$$y=(p^2+k)x.$$

The signed area enclosed by curve and chord is

$$\begin{aligned} &\int_{-p}^{p} x^3+kx-(p^2+k)x\,dx \\ =& \int_{-p}^{p} x^3-p^2x\,dx \\ =& \left[\tfrac{1}{4}x^4-\tfrac{1}{2}p^2x^2\right]_{-p}^{p} \\ =& \ 0, \text{ as required.} \end{aligned}$$

Since the combined signed area is 0, the areas of the two regions enclosed must be equal. QED.

4071. Using a combinatorics method, the possibility space contains ${}^{52}C_4$ hands. For successful hands, the suits are AABC. There are 4 ways of choosing A, then ${}^{3}C_2$ ways of choosing B, C. Having chosen the suits AABC, there are

$${}^{13}C_2\cdot{}^{13}C_1\cdot{}^{13}C_1$$

ways of choosing the cards. This gives

$$\begin{aligned} p &= \frac{4\cdot{}^{3}C_2\cdot{}^{13}C_2\cdot{}^{13}C_1\cdot{}^{13}C_1}{{}^{52}C_4} \\ &= 0.584 \text{ (3sf)}. \end{aligned}$$

— Alternative Method —

Using a conditioning method, writing P for Paired and S for Solo, the probability of suits PPSS in that order is

$$1\cdot\tfrac{12}{51}\cdot\tfrac{39}{50}\cdot\tfrac{26}{49}.$$

The number of orders of PPSS is ${}^{4}C_2$. Put together,

$$p=1\cdot\tfrac{12}{51}\cdot\tfrac{13}{50}\cdot\tfrac{13}{49}\cdot{}^{4}C_2=0.584 \text{ (3sf)}.$$

4072. At time t, the velocity is $v = u - gt$. So, the kinetic energy is given by

$$\begin{aligned} T &= \tfrac{1}{2}mv^2 \\ &= \tfrac{1}{2}m(u-gt)^2. \end{aligned}$$

Taking the baseline for gravitational potential as the initial position, the vertical displacement is $h = ut - \frac{1}{2}gt^2$. So, gravitational potential is

$$\begin{aligned} V &= mgh \\ &= mgut - \tfrac{1}{2}mg^2t^2. \end{aligned}$$

The total energy is given by

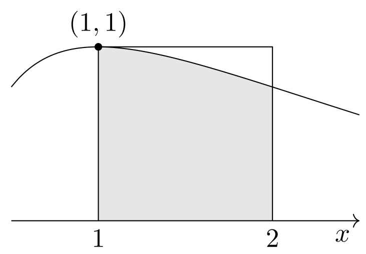
$$\begin{aligned} &T + V \\ &= \tfrac{1}{2}m(u-gt)^2 + mgut - \tfrac{1}{2}mg^2t^2 \\ &\equiv \tfrac{1}{2}m(u^2 - 2ugt + g^2t^2 + 2ugt - g^2t^2) \\ &\equiv \tfrac{1}{2}mu^2. \end{aligned}$$

Hence, the total energy is constant, being equal to $\frac{1}{2}mu^2$, which is the initial kinetic energy with which the projectile was thrown. □

4073. Differentiating by the chain rule,

$$\begin{aligned} & y = \cos(\ln x) \\ \Longrightarrow\ & \frac{dy}{dx} = \frac{-\sin(\ln x)}{x}. \end{aligned}$$

At the point $(1, 1)$, $y = \cos(\ln x)$ has gradient 0. Furthermore, since $\ln x > 0$ for $x \in (1, 2]$, $\frac{dy}{dx} < 0$ over this domain. So, the curve $y = \cos(\ln x)$ is at or below the tangent line $y = 1$.



Therefore, the value of the integral in question (the area of the region shaded above) is less than the area of the marked square. Hence,

$$\int_1^2 \cos(\ln x)\,dx < 1, \text{ as required.}$$

4074. (a) The boundary cases are as follows:

- If $X \subset Y \subset Z$, then the intersection has probability 0.4.
- If X and Y are mutually exclusive, then the intersection has probability 0.

So, $\mathbb{P}(X \cap Y \cap Z)$ has range $[0, 0.4]$.

(b) The boundary cases are as follows:

- If $X \subset Y \subset Z$, then X guarantees both Y and Z.
- If Y and Z are mutually exclusive, then, regardless of whether X has happened, the intersection has probability 0.

So, $\mathbb{P}(Y \cap Z \mid X)$ has range $[0, 1]$.

4075. (a) i. The vertex of the parabola $y = \lambda x(1-x)$ occurs at $x = \frac{1}{2}$, $y = \lambda/4$. So, the range of the function is $[0, \lambda/4]$.

ii. We are told that $P_n \in [0, 1]$ for all $n \in \mathbb{N}$. For this to be true, the maximum cannot exceed 1. This requires $\lambda/4 \le 1$, i.e. $\lambda \le 4$.

(b) i. For fixed points,

$$\begin{aligned} & P = \lambda P(1-P) \\ \Longrightarrow\ & P(1 - \lambda + \lambda P) = 0 \\ \Longrightarrow\ & P = 0, 1 - \tfrac{1}{\lambda}. \end{aligned}$$

ii. If $0 < \lambda < 1$, then $1/\lambda > 1$. Hence, $1 - 1/\lambda < 0$. But population size cannot be negative. Hence, the only fixed point (stable population) is $P = 0$, representing extinction.

iii. Since $\lambda \in [0, 4]$, the value of $1 - 1/\lambda$ is at its greatest when $\lambda = 4$. This gives a fixed point at $P = 3/4$. Hence, the population cannot be stable at $P > 3/4$.

4076. The binomial expansion gives, for $|x| < \frac{1}{2}$,

$$\begin{aligned} \mathrm{f}(x) &= (1-2x)^{-\frac{1}{2}} \\ &= 1 + \left(-\tfrac{1}{2}\right)(-2x) + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2!}(-2x)^2 + ... \\ &\equiv 1 + x + \tfrac{3}{2}x^2 + ... \end{aligned}$$

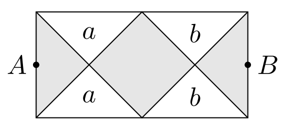
Substituting $x = \frac{1}{32}$,

$$\left(1 - \tfrac{1}{16}\right)^{-\frac{1}{2}} \approx 1 + \tfrac{1}{32} + \tfrac{3}{2}\left(\tfrac{1}{32}\right)^2.$$

Calculating values, $\frac{4}{\sqrt{15}} \approx \frac{2115}{2048}$. This gives

$$\sqrt{15} \approx \tfrac{8192}{2115}, \text{ as required.}$$

4077. There are 2^7 outcomes in the possibility space. In successful outcomes, the regions which border A and B must be shaded, and the central region must be shaded.



At least one of the two regions marked a must be shaded, and at least one of the regions marked b must be shaded. This gives 3 options for the a's and 3 options for the b's. So, the probability p that A and B are connected by a contiguous region is

$$p = \frac{3 \times 3}{2^7} = \tfrac{9}{128}.$$

4078. (a) The cylinder's cross-section is an (x, y) circle, centre $(2, 6)$, radius. So, the axis of symmetry is the line $x = 2$, $y = 6$, extending in the z direction.

i. $a_1 = 2$, $a_2 = 6$, $b_1 = 0$, $b_2 = 0$.

ii. The axis of symmetry is parallel to the z axis. Hence, increasing the parameter t must move the point in the z direction. If $b_3 = 0$, this will not happen.

iii. a_3 is the z value of the "starting point" of the line. The effect of changing a_3 is to translate the line in the z direction, which does nothing.

(b) The (x, y) cross-sectional area is 4π, and the z length is 5. So, the volume is 20π.

4079. Let $\theta = \frac{1}{2}x$. Using two versions of the $\cos 2\theta$ double-angle formula, the LHS may be simplified

$$\sqrt{\frac{1-\cos 2\theta}{1+\cos 2\theta}} \equiv \sqrt{\frac{2\sin^2\theta}{2\cos^2\theta}} \equiv \sqrt{\tan^2\theta}.$$

Since $\tan\theta > 0$, this is equal to $\tan\theta$, which we can rewrite as $\tan\frac{1}{2}x$. □

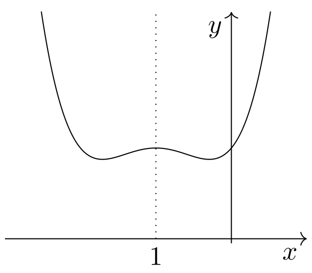
4080. We are looking for a line of symmetry $x = k$ of the quartic $y = \mathrm{f}(x)$. For SPs,

$$\begin{aligned} &4x^3 + 12x^2 + 10x + 2 = 0 \\ \Longrightarrow\ & x = -1, -1 \pm \tfrac{\sqrt{2}}{2}. \end{aligned}$$

These are symmetrical in $x = -1$. So, we need to show symmetry around this value. Expressing $\mathrm{f}(x)$ as a polynomial in $(x + 1)$,

$$\begin{aligned} &x^4 + 4x^3 + 5x^2 + 2x + 2 \\ \equiv\ & (x+1)^4 - (x+1)^2 + 2. \end{aligned}$$

Since all powers are even, $y = \mathrm{f}(x)$ has $x = k$ as a line of symmetry.



Therefore, $\mathrm{f}(-1-x) \equiv \mathrm{f}(-1+x)$, as required.

4081. The vectors are

$$\overrightarrow{AC} = \begin{pmatrix} 29-q \\ -q^2-q \end{pmatrix}, \quad \overrightarrow{BC} = \begin{pmatrix} 29-q^2 \\ -q \end{pmatrix}.$$

Since $\overrightarrow{AC} = k\overrightarrow{BC}$, the vectors are parallel, so their gradients are the same:

$$\begin{aligned} \frac{-q^2-q}{29-q} &= \frac{-q}{29-q^2} \\ \Longrightarrow (q^2+q)(29-q^2) &= q(29-q) \\ \Longrightarrow q^2(q-5)(q+6) &= 0 \\ \Longrightarrow q &= 0, 5, -6. \end{aligned}$$

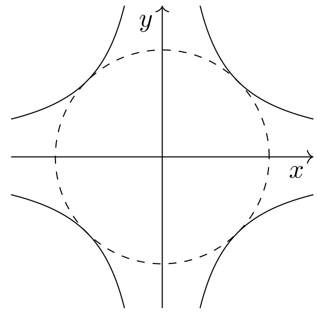
All three values satisfy the conditions.

4082. Adding the first and third equations, $2x = 4$, so $x = 2$. The first two equations become

$$\begin{aligned} y + z &= 2 \\ 2y + z &= 7. \end{aligned}$$

Solving these gives $x = 2$, $y = 5$, $z = -3$.

4083. Consider the curve $x^2y^2 = 1$. This is $xy = \pm 1$, consisting of positive and negative versions of the standard reciprocal graph.

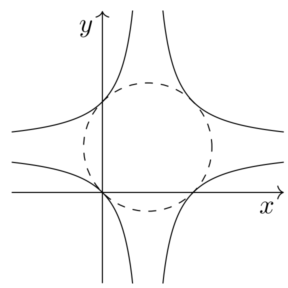


By symmetry, the relevant circle (dashed) must be tangent to this curve at $(\pm 1, \pm 1)$. So, the circle has radius $\sqrt{2}$. Its equation is

$$x^2 + y^2 = 2.$$

We then transform the entire problem, replacing x by $2x - 1$ and y by $2y - 1$. This takes $x^2y^2 = 1$ onto $(2x-1)^2(2y-1)^2 = 1$, and $x^2 + y^2 = 2$ onto

$$(2x-1)^2 + (2y-1)^2 = 2.$$



4084. (a) Under T_1, $p\mathbf{i} + q\mathbf{j}$ transforms to $3p\mathbf{i} + q\mathbf{j}$. Then, under T_2, this transforms to $3p\mathbf{i} + 4q\mathbf{j}$.

(b) If vector $\mathbf{a}$ is unchanged, then $\mathbf{b} = \mathbf{a}$. This is $3p\mathbf{i} + 4q\mathbf{j} = p\mathbf{i} + q\mathbf{j}$. Equating coefficients of $\mathbf{i}$, $p = 0$ and of $\mathbf{j}$, $q = 0$. Hence, only the zero vector is unaffected by the composition of T_1 and T_2.

(c) i. The composition sends $(1,0)$ to $(3,0)$ and $(0,1)$ to $(0,4)$. The length scale factors, from the origin, are 3 and 4. These are different. So, the composition cannot be a single two-dimensional enlargement.

ii. Any one-dimensional stretch must leave one dimension invariant. But, from part (b), the composition of T_1 and T_2 leaves no points other than the origin invariant. Hence, the composition cannot be a single one-dimensional stretch.

4085. The exact positions of the points aren't important. Only their order around the circumference is. So, consider the possibility space as the $4! = 24$ orders of $ABCD$. Of these, successful outcomes alternate between the pair AB and the pair CD. There are 4 choices for the first letter, then 2 for the second, 1 for the third and 1 for the fourth. This gives

$$p = \frac{4 \cdot 2 \cdot 1 \cdot 1}{24} = \tfrac{1}{3}.$$

— Alternative Method —

The exact positions of the points don't matter, so we can place points A, B, C (allowing rotations and reflections) without loss of generality.

This leaves three positions (the gaps AB, BC and AC) in which point D could go. For success, D must be placed away from C, i.e. between A and B. This is one position (in the order) out of three possible positions. So, $p = 1/3$.

4086. Let $u_0 = k$. Noting that $k \notin \{0, 1\}$,

$$\begin{aligned} u_0 &= k, \\ u_1 &= 1 - \tfrac{1}{k} \equiv \tfrac{k-1}{k}, \\ u_2 &= 1 - \frac{1}{\frac{k-1}{k}} \equiv 1 - \tfrac{k}{k-1} = \tfrac{-1}{k-1} \\ u_3 &= 1 - \frac{1}{\frac{-1}{k-1}} \equiv 1 + k - 1 \equiv k. \end{aligned}$$

Since $u_3 = u_0$, and since u_{n+1} depends only on u_n, the sequence is periodic, with period 3.

4087. Consider generic stretches by scale factor p in the x direction and by q in the y direction, applied to $y = x^4 - x^2$:

$$y = q\left(\left(\tfrac{x}{p}\right)^4 - \left(\tfrac{x}{p}\right)^2\right).$$

To produce the required curve,

$$\tfrac{q}{p^4}x^4 - \tfrac{q}{p^2}x^2 \equiv ax^4 - 4ax^2.$$

Equating coefficients, $q/p^4 = a$ and $q/p^2 = 4a$. We divide these: $p^2 = 4a/a = 4$, so $p = 2$ and $q = 16a$.

Hence, the curve has been stretched by LSF 2 in the x direction and by LSF $16a$ in the y direction. The area scale factor is $32a$.

4088. (a) Horizontally and vertically,

$$\begin{aligned} 10 &= (20\cos\theta)t, \\ 2 &= (20\sin\theta)t - 5t^2. \end{aligned}$$

Substituting the former into the latter, then using the second Pythagorean trig identity,

$$\begin{aligned} & 2 = 20\tan\theta - \frac{5}{4\cos^2\theta} \\ \implies & 2 = 10\tan\theta - \tfrac{5}{4}(1 + \tan^2\theta) \\ \implies & 8 = 40\tan\theta - 5 - 5\tan^2\theta \\ \implies & 5\tan^2\theta - 40\tan\theta + 13 = 0. \end{aligned}$$

(b) Solving the quadratic in $\tan\theta$,

$$\begin{aligned} & \tan\theta = 4 \pm \sqrt{\tfrac{67}{5}} \\ \implies & \theta = 18.7°, 82.6° \text{ (1dp)}. \end{aligned}$$

4089. We need to show that $a^2 + b^2$ and $a^2 + c^2$ are both perfect squares. That $b^2 + c^2$ is a perfect square then follows by symmetry.

Firstly,

$$\begin{aligned} & a^2 + b^2 \\ = {} & p^2\left(4q^2 - r^2\right)^2 + q^2\left(4p^2 - r^2\right)^2 \\ \equiv {} & 16p^2q^4 - 8p^2q^2r^2 + p^2r^4 \\ & \quad + 16p^4q^2 - 8p^2q^2r^2 + q^2r^4 \\ \equiv {} & 16p^2q^2\left(p^2 + q^2\right) - 16p^2q^2r^2 + \left(p^2 + q^2\right)r^4. \end{aligned}$$

Since (p, q, r) is a Pythagorean triple, $p^2 + q^2 = r^2$. This means that the first two terms cancel, leaving $r^6 \equiv (r^3)^2$, a perfect square.

Secondly,

$$\begin{aligned} & a^2 + c^2 \\ = {} & 16p^2q^4 - 8p^2q^2r^2 + p^2r^4 + 16p^2q^2r^2 \\ \equiv {} & 16p^2q^4 + 8p^2q^2r^2 + p^2r^4 \\ \equiv {} & p^2\left(4q^2 + r^2\right)^2. \end{aligned}$$

This is also a perfect square. By symmetry, $b^2 + c^2$ is too. So, the cuboid (a, b, c) is an Euler brick, as required.

4090. If $\tan\frac{1}{2}x > 0$, then the RHS of the result is well defined. Differentiating it by the chain rule,

$$\begin{aligned} & \frac{d}{dx}\left(\ln\left(\tan\tfrac{x}{2}\right) + c\right) \\ = {} & \frac{\sec^2\frac{x}{2}}{2\tan\frac{x}{2}} \\ \equiv {} & \frac{1}{2\sin\frac{x}{2}\cos\frac{x}{2}}. \end{aligned}$$

Using $\sin 2\theta \equiv 2\sin\theta\cos\theta$, this is $\frac{1}{\sin x}$, which is $\operatorname{cosec} x$. So,

$$\int \operatorname{cosec} x \, dx = \ln\left(\tan\tfrac{x}{2}\right) + c, \text{ as required.}$$

4091. Assume, for a contradiction, that prime numbers $p_1 < p_2 < p_3$ form a GP. Equating the ratios,

$$\frac{p_2}{p_1} = \frac{p_3}{p_2}$$
$$\Longrightarrow p_2^2 = p_1 p_3.$$

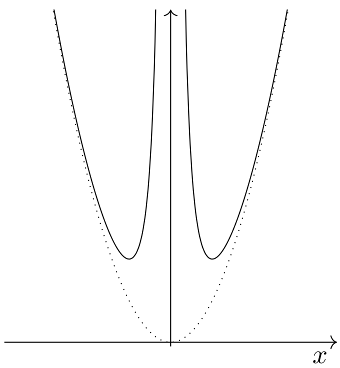
The LHS has no prime factors other than p_2. So, p_1 and p_3 must be equal to p_2. This contradicts the fact that $p_1 < p_2 < p_3$. Therefore, three distinct primes cannot form a GP. □

4092. Once a random number n is chosen, the number of X_i variables which then change, V, is distributed binomially as $V \sim B(n, 1/2)$. The probabilities are

n	$\mathbb{P}(V = 2)$
0	0
1	0
2	${}^2\text{C}_2 \cdot \frac{1}{4} = \frac{1}{4}$
3	${}^3\text{C}_2 \cdot \frac{1}{8} = \frac{3}{8}$
4	${}^4\text{C}_2 \cdot \frac{1}{16} = \frac{3}{8}$.

So, $\mathbb{P}(\text{two change}) = \frac{1}{5} \cdot \frac{1}{4} + \frac{1}{5} \cdot \frac{3}{8} + \frac{1}{5} \cdot \frac{3}{8} = \frac{1}{5}$.

4093. As $x \to \pm\infty$, the right-hand term tends to zero, so, for large x, the curve tends towards the parabola $y = x^2$ (dotted below). Also, there is a double asymptote at $x = 0$, with $y \to +\infty$ as $x \to 0^+$ and as $x \to 0^-$:



The minima are at $(\pm 1, 2)$.

4094. The number of squares threatened depends on proximity to the side of the board. The number of squares threatened is as follows:

2	3	4	4	4	4	3	2
3	4	6	6	6	6	4	3
4	6	8	8	8	8	6	4
4	6	8	8	8	8	6	4
4	6	8	8	8	8	6	4
4	6	8	8	8	8	6	4
3	4	6	6	6	6	4	3
2	3	4	4	4	4	3	2

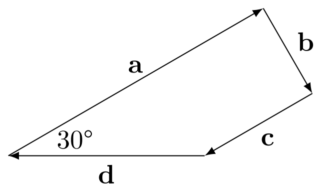
Conditioning on the number of squares threatened (starting at 2), the probability p that the knights threaten each other is

$$\tfrac{4}{64} \cdot \tfrac{2}{63} + \tfrac{8}{64} \cdot \tfrac{3}{63} + \tfrac{20}{64} \cdot \tfrac{4}{63} + \tfrac{16}{64} \cdot \tfrac{6}{63} + \tfrac{16}{64} \cdot \tfrac{8}{63}.$$

This gives $p = \frac{1}{12}$, as required.

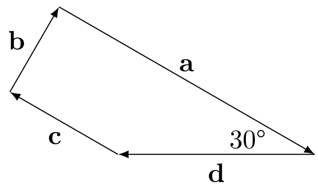
4095. The four vectors sum to zero; so, when placed tip-to-tail, they must form a closed quadrilateral. Since **a** and **c** are parallel, they must form opposite edges. Likewise **b** and **d**.

Resolve in the direction of **b**. Since **a** and **c** have no component in this direction, the component of **d** in the direction of **b** must have length 1. This gives two possible configurations, in which **a** and **c** may have arbitrary (linked) lengths. The first is



$$\mathbf{b} = \tfrac{1}{2}\mathbf{i} - \tfrac{\sqrt{3}}{2}\mathbf{j}$$

The second configuration is



$$\mathbf{b} = \tfrac{1}{2}\mathbf{i} + \tfrac{\sqrt{3}}{2}\mathbf{j}$$

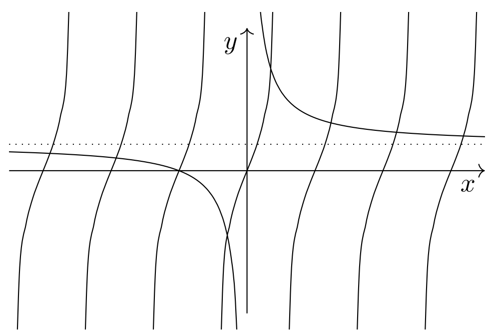
4096. Using the quotient rule,

$$\frac{dy}{dx} = \frac{\cos x(1 + x^2) - \sin x(2x)}{(1 + x^2)^2}.$$

For SPs,

$$\cos x(1 + x^2) - \sin x(2x) = 0$$
$$\Longleftrightarrow 2\tan x = 1 + \tfrac{1}{x}.$$

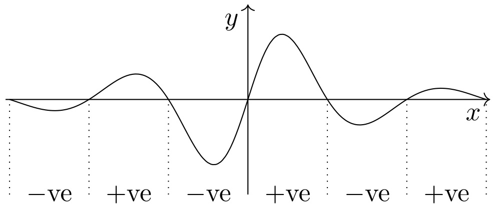
Consider $y = 2\tan x$ and $y = 1 + \frac{1}{x}$. The latter approaches $y = 1$ as $x \to \infty$ and the former is periodic, as shown:



The curves intersect infinitely many times. So, the original curve has infinitely many SPs.

—— Alternative Method ——

The denominator $1 + x^2$ has no real roots. So, the curve has no discontinuities, and is well defined over $\mathbb{R}$. Furthermore, the denominator is always positive. The numerator has infinitely many roots, which are at $x = n\pi$, for $n \in \mathbb{Z}$. On domains of the form $\big(2k\pi, (2k+1)\pi\big)$, y is positive; on domains of the form $\big((2k+1)\pi, (2k+2)\pi\big)$, y is negative.



So, there must be a local maximum in every +ve domain $\big(2k\pi, (2k+1)\pi\big)$, and a local minimum in every −ve domain $\big((2k+1)\pi, (2k+2)\pi\big)$. Hence, the curve has infinitely many SPs, as required.

4097. Since f is quartic, $y = \text{f}(x)$ must have at least one turning point. A polynomial function f cannot be invertible over an interval containing a turning point, so $x = \alpha$ must be a turning point.

We also know that $\text{f}(\alpha) = 0$. Since $x = \alpha$ is both a root and a stationary point of $\text{f}(x)$, it must be a repeated root. Hence, $\text{f}(x)$ must have a repeated factor of $(x - \alpha)$, i.e. a factor of $(x - \alpha)^2$. □

4098. Multiplying by e^x and rearranging, the equation is

$$e^{4x} + e^{2x} + 1 = 0.$$

This is a quadratic in e^{2x}. Its discriminant is $\Delta = -3 < 0$, so it has no real roots. Hence, no $x \in \mathbb{R}$ satisfies the given equation.

4099. Let $y = \text{f}(x)$. Separating the variables,

$$\begin{aligned} & x\frac{dy}{dx} = (x+1)y^2 \\ \Longrightarrow & \int y^{-2}\, dy = \int 1 + \frac{1}{x}\, dx \\ \Longrightarrow & -y^{-1} = 1 + x + \ln|x| + c \\ \Longrightarrow & \, y^{-1} = b - x - \ln|x| \\ \Longrightarrow & \, 1 = y\big(b - x - \ln|x|\big). \end{aligned}$$

Reintroducing $\text{f}(x)$, this is

$$\text{f}(x)\big(b - x - \ln|x|\big) = 1.$$

This is the general solution to the DE, so it holds identically, for any x. We can substitute $x = a$, giving the required result:

$$\text{f}(a)\big(b - a - \ln|a|\big) = 1.$$

4100. Treating the 10 letters as distinguishable, there are 10! orders of the letters. However, in this list of 10! arrangements, we overcount, due to the repeated letters. So, we need to divide by an overcounting factor of 2! for the orders of L_1L_2, 2! for the orders of E_1E_2, and 2! for the orders of S_1S_2. This gives

$$\frac{10!}{2! \times 2! \times 2!} = 453600, \text{ as required.}$$

—— End of 41st Hundred ——